On a Class of Cubic Surfaces with Curves of the Same Species.

By John Eiesland.

I.

"In Raume (x, y, z)," says Lie,* "kennen wir noch einige Flächen, die unendlich viele Scharen von Curven gleicher Gattung enthalten, allerdings nicht so viele wie die oben besprochene Flächenart." By "die oben besprochene Flächenart" he refers to the tetrahedral symmetrical surfaces

$$x^{\alpha}y^{\alpha}z^{\alpha} = \rho$$

which contain ∞^2 families of curves of the same species.† He then mentions two kinds of surfaces having the same property, viz.: the planes

$$Ax + By + Cz + D = 0$$

and the ∞^5 quadric surfaces

$$Ayz + Bxz + Cxy + Lx + My + Nz = 0,$$

the latter having 4 families, namely the rectilinear generators and two families of twisted cubics. In the following I have used an indirect method of obtaining other surfaces of the same kind, no direct method being obvious. The method consists in first finding translation-surfaces of a certain form that correspond to a unicursal quartic, irreducible or not, in the plane at infinity, and then transforming these to the so-called logarithmic space by means of the transformation

$$X = \log x, \qquad Y = \log y, \qquad Z = \log z, \tag{1}$$

the space (X, Y, Z) being the space of the translation-surface. This transformation has been used by Lie for this and other purposes.‡

* Lie-Scheffers, "Geometrie der Berührungstransformationen," p. 363.

† For definition of the term "species" in the sense here used, see *ibid.*, pp. 333, 334.

‡ Lie-Scheffers, "Berührungstransformationen," p. 356.

Lie has shown that to all reducible quartic curves in the plane at infinity consisting of two intersecting conics there correspond translation-surfaces of the form

$$Ae^{Y+Z}+Be^{Z+X}+Ce^{X+Y}+Le^{X}+Me^{Y}+Ne^{Z}=0, \tag{2}$$

and all their transforms by a linear projective transformation. The points P and P', where the two tangents at the points of intersection of the two conjugate translation-curves pierce the plane at infinity, must be taken on different conics if the surface shall have a fourfold mode of generation. If P and P' are taken on the same conic we get surfaces that can be generated in ∞ number of ways.

By means of the logarithmic transformation (1) the surface (2) is transformed into a quadric

$$Axy+Bzx+Cxy+Lx+My+Nz=0, \tag{3}$$

which has four families of *curves of the same species*,* corresponding to the four families of translation-curves. (See Lie–Scheffers, "Berührungstransformationen," p. 347). These curves consist of two sets of rectilinear generators and two families of twisted cubics which pass through the vertices of the tetrahedron of reference.

Since the surface (3) contains no edge of the tetrahedron, there exists an involutory transformation

$$x_1=\frac{\lambda}{x},\quad y_1=\frac{\mu}{y},\quad z_1=\frac{\nu}{z} \tag{4}$$

for which the surface remains invariant; the two families of cubic curves are transformed into the two sets of twisted cubics and *vice versa*. For proof of this and other theorems we refer the reader to the Lie–Scheffers volume on contact transformations.

In a paper published in Vol. XXX of AM. JOUR. OF MATH.† I have extended the theory of translation-surfaces to the more general case in which the curve in the plane at infinity is an irreducible unicursal quartic. It was found that to such a curve, having real double points with distinct tangents, there corresponds a translation-surface of the form

$$A+Be^{X}+Ce^{Y}+De^{Z}+Ee^{X+Z}+Fe^{X+Y}+Ge^{Z+Y}+He^{X+Y+Z}=0 \tag{5}$$

with the following fundamental relation between the coefficients:

$$EGAF=HDCB, \tag{6}$$

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*To a translation in the space (X, Y, Z) there corresponds a transformation of the form $x_1=\lambda x$, $y_1=\mu y$, $z_1=\nu z$; hence, to the translation-curves belonging to any one family there corresponds a family of curves of the same species, using Lie's definition in "Theorie der Berührungstransformationen," p. 330.

† "On Translation-Surfaces Connected with a Unicursal Quartic," Vol. XXX, pp. 170–208.

which expresses the property that all surfaces (5) have a *center of symmetry.* Moreover, since there are ∞^3 unicursal quartics (projectively non-equivalent), there will be ∞^3 *types* of translation-surfaces (5) that have a fourfold mode of generation. We shall repeat here a few of the formulas obtained in the above-mentioned paper.

If the origin be taken as center of symmetry, the surface takes the form

$$A(1-e^{X+Y+Z})+B'(e^{Y+Z}-e^{X})+C'(e^{X+Z}-e^{Y})+D'(e^{X+Y}-e^{Z})=0. \tag{5'}$$

The parametric representation of (5) is

$$X=\log\frac{(\rho_1-\alpha_1)(\rho_2-\alpha_1)}{(\rho_1-\beta_1)(\rho_2-\beta_1)},\qquad Y=\log\frac{(\rho_1-\alpha_2)(\rho_2-\alpha_2)}{(\rho_1-\beta_2)(\rho_2-\beta_2)},$$
$$Z=\log(\rho_1-k)(\rho_2-k), \tag{5''}$$

where $\alpha_1,\beta_1;\ \alpha_2,\beta_2$ are the roots of the equations

$$\rho_1^2+2b\rho+1=0,$$
$$\rho_1^2+(4b+2mc)\rho_1+m^2+4b^2+4bmc=0$$

respectively, and $k=\dfrac{-2(b+mc)}{1-m^2}$, m being any one root of the equation

$$m^2+2am+1=0;$$

a, b, c are the parameters of the quartic curve

$$x^2+y^2-2axy+x^2y^2-2bx^2y-2cy^2=0. \tag{7}$$

The coefficients $A, B, \ldots, H$ have the following values (AM. JOUR., Vol. XXX, p. 175):

$$\begin{aligned} &A=(\alpha_1-\alpha_2)(k-\alpha_1)(k-\alpha_2), && B=(\alpha_2-\beta_1)(k-\alpha_2)(k-\beta_1), \\ &C=(\beta_2-\alpha_1)(k-\alpha_1)(k-\beta_2), && D=\alpha_2-\alpha_1, \qquad E=\beta_1-\alpha_2, \\ &F=(\beta_1-\beta_2)(k-\beta_1)(k-\beta_2), && G=\alpha_1-\beta_2, \qquad H=\beta_2-\beta_1. \end{aligned} \tag{8}$$

If now we apply the transformation (1) to the surface (5) we obtain a cubic surface,

$$A+Bx+Cy+Dz+Exz+Fxy+Gyz+Hxyz=0, \tag{9}$$

which is analogous to the surface (3) obtained by Lie. We propose to study these surfaces and their characteristic properties.

THEOREM I. *Given any surface* (9) *with the identical relation* $AEGF=BCDH$ *between the coefficients, there exists an involutory transformation* (4) *which will leave the surface invariant.*

Proof. Transforming we have

$$Axyz+B\lambda yz+C\mu xz+D\nu yx+E\lambda\nu y+F\lambda\mu z+G\nu\mu x+H\lambda\mu\nu=0,$$

which will be identical with (9), if we put

$$\lambda = \frac{AG}{BH}, \quad \mu = \frac{EA}{CH}, \quad \nu = \frac{FA}{DH}, \tag{10}$$

taking also account of the relation (6). Q. E. D.

The point $(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu})$ we shall call the *center of involution;* in fact, to an involution in space x, y, z there corresponds in (X, Y, Z) an inversion (Spieglung) and, since the center of symmetry is midway between the points X, Y, Z and $-X, -Y, -Z$, in the space x, y, z, the corresponding center of involution must be $(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu})$.

Consider the transformation

$$x = \lambda' x_1, \quad y = \mu' y_1, \quad z = \nu' z_1. \tag{11}$$

All surfaces that are transforms of (9) by this transformation we shall call equivalent. In particular, it is possible to find an equivalent surface whose center of involution is $(-1, -1, -1)$, in which case it is of the form

$$A(1 + xyz) + B'(yz + x) + C'(xz + y) + D'(xy + z) = 0; \tag{12}$$

in fact, if we perform the transformation, we have

$$A + B\lambda' x_1 + C\mu' y_1 + D\nu' z_1 + E\lambda'\nu' x_1 z_1 + F\lambda'\mu' x_1 y_1 + G\mu'\nu' y_1 z_1 + H\lambda'\mu'\nu' x_1 y_1 z_1 = 0, \tag{13}$$

and putting $A = H\lambda'\mu'\nu'$, $B\lambda' = G\nu'\mu'$, $C\mu' = E\lambda'\nu'$, $D\nu' = F\lambda'\mu'$, we have, using the identity (6),

$$\lambda' = \sqrt{\frac{AG}{BH}}, \quad \mu' = \sqrt{\frac{EA}{CH}}, \quad \nu' = \sqrt{\frac{FA}{DH}}. \tag{14}$$

Substituting these values in (13) we have

$$A(1 + xyz) + B'(yz + x) + C'(xz + y) + D'(xy + z) = 0, \tag{12}$$

where

$$B' = B\sqrt{\frac{AG}{BH}}, \quad C' = C\sqrt{\frac{EA}{CH}}, \quad D' = D\sqrt{\frac{FA}{DH}}.$$

This equation might also have been obtained from (9) by putting

$$x = \frac{\sqrt{\lambda}}{x'}, \quad y = \frac{\sqrt{\mu}}{y'}, \quad z = \frac{\sqrt{\nu}}{z'}.$$

Hence, to the involutory center $(-1, -1, -1)$ of (12) corresponds the center $(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\nu})$ of (9) as found above.

If we transform (9) to the form

$$A(1-xyz)+B'(yz-x)+C'(xz-y)+D'(xy-z)=0,$$

which has the center of involution (1, 1, 1), the involutory transformation is

$$x=\frac{-\sqrt{\lambda}}{x'},\quad y=\frac{-\sqrt{\mu}}{y'},\quad z=-\frac{\sqrt{\nu}}{z'}.$$

We have thus proved:

THEOREM II. *It is always possible by means of a transformation* (11) *to transform the surface* (9) *into an equivalent surface of the form*

$$A(1+xyz)+B(yz+x)+C(xz+y)+D(xy+z)=0.$$

Theorems I and II are true for any surface of the form (9) provided the identical relation $EGAF=HDCB$ holds. On account of some unimportant but special assumptions made in deriving the surface (5) it appears from the equations (8) that two other identical relations exist. One of these is $D+E+G+H=0$, the second, also homogeneous, is easily found but may be omitted here. Both are accidental and have no special geometric significance. The relation $EGAF=BCDH$, on the other hand, is fundamental, as was observed before, p. 2. If we assume, *à priori*, the coefficients $A, B, \ldots, H$ arbitrary and real, we can not say that the parametric representation of (5) is expressed by the equations (5″). It will be shown later how to obtain the parametric representation of (9) when only (6) is true.

Let us consider the surface (9), whose coefficients have the values given by (8). The parametric representation is obtained from (5″) as follows:

$$x=\frac{(\rho_1-\alpha_1)(\rho_2-\alpha_1)}{(\rho_1-\beta_1)(\rho_2-\beta_1)},\quad y=\frac{(\rho_1-\alpha_2)(\rho_2-\alpha_2)}{(\rho_1-\beta_2)(\rho_2-\beta_2)},\quad z=(\rho_1-k)(\rho_2-k). \tag{16}$$

If now we use the transformation

$$x=\frac{\lambda}{x_1},\quad y=\frac{\mu}{y_1},\quad z=\frac{\nu}{z_1}, \tag{4}$$

where λ, μ, ν have the values

$$\lambda=\frac{AG}{BH},\quad \mu=\frac{EA}{CH},\quad \nu=\frac{FA}{DH}, \tag{10}$$

we obtain the same surface (16) but with a different mode of representation, viz.:

$$x_1 = \frac{\lambda(\rho_3 - \beta_1)(\rho_4 - \beta_1)}{(\rho_3 - \alpha_1)(\rho_4 - \alpha_1)}, \quad y_1 = \frac{\mu(\rho_3 - \beta_2)(\rho_4 - \beta_2)}{(\rho_3 - \alpha_2)(\rho_4 - \alpha_2)}, \quad z_1 = \frac{\nu}{(\rho_3 - k)(\rho_4 - k)}. \tag{16'}$$

ρ_1, ρ_2, ρ_3 and ρ_4 are the four families of twisted cubics corresponding to the four translation-curves on (5). The involutory transformation (4) has transformed ρ_1 and ρ_2 into ρ_3 and ρ_4, and, if we start with (16′), ρ_3 and ρ_4 into ρ_1 and ρ_2. It may also be observed that the parametric representation of (12) may be obtained by putting $\sqrt{\lambda}$, $\sqrt{\mu}$, $\sqrt{\nu}$ for λ, μ, ν in (16′) so that (12) may be put in either one of the forms

$$x = \frac{\sqrt{\lambda}(\rho_3 - \beta_1)(\rho_4 - \beta_1)}{(\rho_3 - \alpha_1)(\rho_4 - \alpha_1)}, \quad y = \frac{\sqrt{\mu}(\rho_3 - \beta_2)(\rho_4 - \beta_2)}{(\rho_3 - \alpha_2)(\rho_4 - \alpha_2)}, \quad z = \frac{\sqrt{\nu}}{(\rho_3 - k)(\rho_4 - k)}; \tag{12'}$$

$$x = \frac{(\rho_1 - \alpha_1)(\rho_2 - \alpha_1)}{\sqrt{\lambda}(\rho_1 - \beta_1)(\rho_2 - \beta_1)}, \quad y = \frac{(\rho_1 - \alpha_2)(\rho_2 - \alpha_2)}{\sqrt{\mu}(\rho_1 - \beta_2)(\rho_2 - \beta_2)}, \quad z = \frac{(\rho_1 - k)(\rho_2 - k)}{\sqrt{\nu}}. \tag{12''}$$

Let there be given a unicursal quartic in the plane at infinity, and let the tangents at the double points be real and distinct. We write the equation, as before,

$$x^2 + y^2 - 2axy + x^2y^2 - 2bx^2y - 2cy^2 = 0. \tag{9}$$

To it there corresponds the translation-surface (5). To the ∞^3 values of the parameters a, b, c correspond ∞^3 *types* of surfaces (5), since the coefficients $A, \ldots, H$ are functions of these three parameters. Using the logarithmic transformation we obtain the cubic surface (9). We may now extend the term *type* also to these transforms of the ∞^3 translation-surfaces and we shall say that they correspond logarithmically to the ∞^3 non-equivalent quartics in the plane at infinity of the (X, Y, Z) space, so that we may say:

To the ∞^3 projectively non-equivalent unicursal quartics with double points having distinct tangents there correspond in the logarithmic space ∞^3 types of cubic surfaces

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0. \tag{9}$$

By employing the ∞^3 transformations

$$x_1 = \lambda' x, \quad y_1 = \mu' y, \quad z_1 = \nu' z, \tag{11}$$

we obtain ∞^3 surfaces corresponding to the ∞^3 translation-surfaces. Therefore, to any given quartic, (a, b, c being fixed parameters) there corresponds in the logarithmic space ∞^3 cubic surfaces. Hence the

THEOREM III. *To all the ∞^3 non-equivalent unicursal quartics in the plane at infinity there correspond in the logarithmic space ∞^6 cubic surfaces* (9). *These arrange themselves in ∞^3 types, ∞^3 surfaces belonging to the same type. The family of ∞^3 surfaces belonging to each type is invariant for the ∞^3 transformations* (11). *For each surface there exists an involutory transformation*

$$x_1 = \frac{\lambda}{x}, \quad y_1 = \frac{\mu}{y}, \quad z_1 = \frac{\nu}{z}, \tag{4}$$

for which the surface remains invariant; the surface contains 4 *families of twisted cubics which group themselves in pairs such that each pair belongs to the same species and one pair is by the transformation* (4) *transformed into the second pair and vice versa.*

II.

We shall now resume the study of the surfaces (9), assuming that the coefficients $A, B, \ldots, H$ satisfy the single identical relation $EGAF = HDCB$. In the first place we notice that it has three nodes at the three vertices of the tetrahedron of reference that are situated in the plane at infinity. In order to begin with the most general case we shall assume these to be ordinary nodes. The three edges joining the nodes will lie on the surface, each counting as four, and will account for 12 of the 27 straight lines on the surface. The surface is of the sixth class, since it has three nodes (Salmon, "Geometry of Three Dimensions," pp. 488, 489, fourth ed.).

Let there now be given a surface

$$x = \frac{(a_1\rho_1 + b_1)(a_1\rho_2 + b_1)}{(c_1\rho_1 + d_1)(c_1\rho_2 + d_1)}, \quad y = \frac{(a_2\rho_1 + b_2)(a_2\rho_2 + b_2)}{(c_2\rho_1 + d_2)(c_2\rho_2 + d_2)},$$

$$z = \frac{(a_3\rho_1 + b_3)(a_3\rho_2 + b_3)}{(c_3\rho_1 + d_3)(c_3\rho_2 + d_3)};$$

introducing new parameters ρ_1', ρ_2' defined by the equations

$$\rho_1' = \frac{a_3\rho_1 + b_3}{c_3\rho_1 + d_3}, \quad \rho_2' = \frac{a_3\rho_2 + b_3}{c_3\rho_2 + d_3},$$

this surface may be put into the form

$$x = \frac{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}{(\gamma_1\rho_1 - \delta_1)(\gamma_1\rho_2 - \delta_1)}, \quad y = \frac{(\rho_1 - \beta_2)(\rho_2 - \beta_2)}{(\gamma_2\rho_1 - \delta_2)(\gamma_2\rho_2 - \delta_2)}, \quad z = \rho_1\rho_2, \tag{17}$$

from which, by eliminating ρ_1 and ρ_2, we get the surface

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \tag{18}$$

where $A, B, \ldots, H$ have the following values:

$$\left.\begin{array}{lll} A = \beta_1\beta_2(\beta_1 - \beta_2). & B = \delta_1\beta_2(\beta_2\gamma_1 - \delta_1), & C = \beta_1\delta_2(\delta_2 - \beta_1\gamma_2), \\ D = \beta_2 - \beta_1, & E = \gamma_1(\delta_1 - \beta_2\gamma_1), & F = \delta_1\delta_2(\delta_1\gamma_2 - \delta_2\gamma_1), \\ G = \gamma_2(\beta_1\gamma_2 - \delta_2), & H = \gamma_1\gamma_2(\gamma_1\delta_2 - \delta_1\gamma_2). & \end{array}\right\} \tag{19}$$

From these equations we easily see that the relation $AEGF = HBCD$ is identically satisfied. *The surface* (17) *therefore belongs to the class of surfaces we are discussing.* The curves ρ_1 and ρ_2 are the twisted cubics of a pair of families belonging to the same species, the second pair, ρ_3 and ρ_4, also of the same species, being obtained by taking the reciprocals of x, y and z and multiplying each by the quantities $\frac{AG}{BH}$, $\frac{EA}{CH}$, $\frac{FA}{DH}$, respectively. (See pp. 5 and 6.)

Conversely, *given a surface* (18) *with real non-vanishing coefficients satisfying the identical relation* (6), *a parametric representation of the form* (17) *may be found.*

In the first place, it is clear, that if in (17) $\beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2$ are all real, the coefficients of (18) will all be real. If, however, (18) be given with real coefficients, it is not to be expected that these quantities will always be real in the parametric representation; but this point will be made clear later. We shall show how to determine (17) when (18) is given. We must have

$$\begin{array}{lll} \rho A = \beta_1\beta_2(\beta_1 - \beta_2), & \rho B = \delta_1\beta_2(\beta_2\gamma_1 - \delta_1), & \rho C = \beta_1\delta_2(\delta_2 - \gamma_2\beta_1), \\ \rho D = \beta_2 - \beta_1, & \rho E = \gamma_1(\delta_1 - \beta_2\gamma_1), & \rho F = \delta_1\delta_2(\delta_1\gamma_2 - \delta_2\gamma_1), \\ \rho G = \gamma_2(\beta_1\gamma_2 - \delta_2), & \rho H = \gamma_1\gamma_2(\gamma_1\delta_2 - \delta_1\gamma_2), & \end{array} \tag{20}$$

where ρ is a factor of proportionality.

This system contains seven independent equations and seven unknowns. We shall show how to solve it. We obtain by division the following simple equations

$$\frac{A}{D} = -\beta_1\beta_2, \quad \frac{B}{E} = -\frac{\delta_1}{\gamma_1}\beta_2, \quad \frac{C}{G} = -\frac{\delta_2}{\gamma_2}\beta_1, \quad \frac{F}{H} = -\frac{\delta_1\delta_2}{\gamma_1\gamma_2}, \tag{21}$$

of which only three are independent owing to the relation (6). We also find from (20) and (21)

$$\frac{D}{B} = -\frac{\frac{A}{D} + \beta_2^2}{\beta_2^2\delta_1^2\left(1 + \frac{E}{B}\,\beta_2^2\right)}, \quad \frac{A}{C} = \frac{-\left(\frac{A}{D} + \beta_2^2\right)\beta_2^2}{\delta_2^2\left(\frac{A^2C}{D^2G} + \beta_2^2\right)},$$

from which we obtain

$$\delta_1^2 = -\frac{\frac{B}{D}\left(\frac{A}{D}+\beta_2^2\right)}{\beta_2^2\left(1+\frac{E}{B}\beta_2^2\right)}, \quad \delta_2^2 = -\frac{\beta_2^2\left(\frac{A}{D}+\beta_2^2\right)}{\frac{A}{C}\left(\beta_2^2+\frac{A^2G}{D^2C}\right)}; \tag{22}$$

from the same equations we have also

$$\gamma_1^2 = -\frac{\frac{E^2}{BD}\left(\frac{A}{D}+\beta_2^2\right)}{1+\frac{E}{B}\beta_2^2}, \quad \gamma_2^2 = -\frac{\frac{AG^2}{CD^2}\left(\frac{A}{D}+\beta_2^2\right)}{\beta_2^2+\frac{A^2G}{D^2C}}. \tag{23}$$

From (20) we have again

$$\frac{D}{F} = \frac{\frac{A}{D}+\beta_2^2}{\delta_1^2\delta_2^2\left(\frac{AG}{DC}+\frac{E}{B}\beta_2^2\right)}, \quad \delta_1^2\delta_2^2 = \frac{\frac{F}{D}\left(\frac{A}{D}+\beta_2^2\right)}{\frac{AG}{DC}+\frac{E}{B}\beta_2^2}. \tag{24}$$

Combining (22) and (24) we have the quadratic equation determining β_2^2,

$$\beta_2^4 + \frac{AB}{ED}\left(\frac{GB+EC-FD-AH}{CB-AF}\right)\beta_2^2 + \frac{A^2GB}{D^2CE} = 0. \tag{25}$$

β_2^2 being found from this equation, γ_1^2, γ_2^2, δ_1^2, δ_2^2 may be determined from equations (21), (22) and (23). We obtain no new parametric representation if we choose $\beta_2 = -\sqrt{\beta_2^2}$; in fact, substituting $-\rho_1$, $-\rho_2$ for ρ_1 and ρ_2 in (17) we obtain the same surface. If β_2^2 is negative, *i. e.*, $\beta_2 = \pm ik$, we still obtain real curves ρ_1 and ρ_2; in fact, it is easily seen from (21) that β_1, $\frac{\delta_1}{\gamma_1}$, $\frac{\delta_2}{\gamma_2}$ are also pure imaginaries. If then we put (17) in the form

$$\gamma_1^2 x = \frac{(\rho_1-\beta_1)(\rho_2-\beta_1)}{\left(\rho_1-\frac{\delta_1}{\gamma_1}\right)\left(\rho_2-\frac{\delta_1}{\gamma_1}\right)}, \quad \gamma_2^2 y = \frac{(\rho_1-\beta_2)(\rho_2-\beta_2)}{\left(\rho_1-\frac{\delta_2}{\gamma_2}\right)\left(\rho_2-\frac{\delta_2}{\gamma_2}\right)}, \quad z = \rho_1\rho_2,$$

and put $\rho_1 = i\rho_1'$, $\rho_2 = i\rho_2'$, the imaginary unit will disappear from the equations. We have therefore only to consider the two roots of (25), this equation being a quadratic in β_2^2. The discriminant is

$$\Delta = \frac{(GB+EC-FD-AH)^2}{(CB-AF)^2} - \frac{4EG}{CB}, \tag{26}$$

and we have the three cases:

1°. $\Delta > 0$. *The roots are real and the curves ρ_1, ρ_2 are real twisted cubics.*

2°. $\Delta < 0$. *The roots are imaginary and the curves ρ_1 and ρ_2 are imaginary cubics;* it should be noticed that the surface is nevertheless real.

3°. $\Delta = 0$. *The roots are equal.* This limiting case we shall discuss later.

The two roots of (25) *will give a double representation of the surface, and the second is obtained from the first precisely as before on pp.* 5 *and* 6. We found that, given a surface

$$x=\frac{(\rho_1-\beta_1)(\rho_2-\beta_1)}{(\gamma_1\rho_1-\delta_1)(\gamma_1\rho_2-\delta_1)},\quad y=\frac{(\rho_1-\beta_2)(\rho_2-\beta_2)}{(\gamma_2\rho_1-\delta_2)(\gamma_2\rho_2-\delta_2)},\quad z=\rho_1\rho_2, \tag{17}$$

we may obtain the same surface using the transformation

$$x_1=\frac{AG}{BH}\cdot\frac{1}{x},\quad y_1=\frac{EA}{CH}\cdot\frac{1}{y},\quad z_1=\frac{AF}{DH}\cdot\frac{1}{z},$$

namely,

$$\begin{gathered}x=\frac{AG}{BH}\cdot\frac{(\gamma_1\rho_3-\delta_1)(\gamma_1\rho_4-\delta_1)}{(\rho_3-\beta_1)(\rho_4-\beta_1)},\quad y=\frac{EA}{CH}\cdot\frac{(\gamma_2\rho_3-\delta_2)(\gamma_2\rho_4-\delta_2)}{(\rho_3-\beta_2)(\rho_4-\beta_2)}\\ z=\frac{AF}{DH}\cdot\frac{1}{\rho_3\rho_4},\end{gathered} \tag{17'}$$

which we shall prove may be deduced from (17), if in this set of equations we substitute for β_1^2 the second root of (25); let us call it $\beta_1'^2$. We have then from (25)

$$\beta_2^2\beta_2'^2=\frac{A^2GB}{D^2CE}, \tag{27}$$

and from the first of equations (21),

$$\beta_1^2\beta_1'^2=\frac{A\,EC}{D^2BG}\cdot \tag{28}$$

From the same equations and from (20) and (23) we have

$$\begin{aligned}\gamma_2^2\gamma_2'^2&=\frac{G^2AH}{D^2CE},\quad \gamma_1^2\gamma_1'^2=\frac{E^2AH}{D^2BG},\\ \delta^2\delta_1'^2&=\frac{BF}{GD},\quad \delta_2^2\delta_2'^2=\frac{CF}{ED}.\end{aligned} \tag{29}$$

Introducing now in (17) the conjugate roots β_1', β_2', γ', γ_2', δ_1', δ_2', we have the surface

$$x=\frac{(\rho_1-\beta_1')(\rho_2-\beta_1')}{(\gamma_1'\rho_1-\delta_1')(\gamma_1'\rho_2-\delta_1')},\quad y=\frac{(\rho_1-\beta_2')(\rho_2-\beta_2')}{(\gamma_2'\rho_1-\delta_2')(\gamma_2'\rho_2-\delta_2')},\quad z=\rho_1\rho_2. \tag{17''}$$

Substituting for β_1', β_2', γ_1', γ_2', δ_1', δ_2' their values in terms of β_1, β_2, γ_1, γ_2, δ_1, δ_2, obtained from (27), (28) and (29), and putting

$$\rho_1=\frac{\sqrt{\dfrac{FA}{DH}}}{\rho_3},\quad \rho=\frac{\sqrt{\dfrac{FA}{DH}}}{\rho_4},$$

we obtain the surface (17′) which was to be proven. We may now state the results obtained as follows:

THEOREM IV. *Given a cubic surface*

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0,$$

with the identical relation $AEGF = BCDH$ between the coefficients, it is always possible to find a parametric representation of the surface of the form

$$x = \frac{(\rho_1 - \beta_1)(\rho_2 - \beta_1)}{(\gamma_1\rho_1 - \delta_1)(\gamma_1\rho_2 - \delta_1)}, \quad y = \frac{(\rho_1 - \beta_2)(\rho_2 - \beta_1)}{(\gamma_2\rho_1 - \delta_2)(\gamma_2\rho_2 - \delta_2)}, \quad z = \rho_1\rho_2, \tag{17}$$

where β_1^2, β_2^2, γ_1^2, γ_2^2, δ_1^2, δ_2^2 are the roots of certain quadratic equations. A second mode of representation is obtained by taking for β_1^2, β_2^2, γ_1^2, γ_2^2, δ_1^2, δ_2^2 their respective conjugate values, so that the surface is also represented by the equations

$$\left.\begin{aligned} x = \frac{AG}{BH}\,\frac{(\gamma_1\rho_3 - \delta_1)(\gamma_1\rho_4 - \delta_1)}{(\rho_3 - \beta_1)(\rho_4 - \beta_1)}, \quad y = \frac{EA}{CH}\,\frac{(\gamma_2\rho_3 - \delta_2)(\gamma_2\rho_4 - \delta_2)}{(\rho_3 - \beta_2)(\rho_4 - \beta_2)}, \\ z = \frac{AF}{DH}\cdot\frac{1}{\rho_3\rho_4}. \end{aligned}\right\} \tag{17′}$$

The involutory transformation

$$x_1 = \frac{AG}{BH}\cdot\frac{1}{x}, \quad y_1 = \frac{EA}{CH}\cdot\frac{1}{y}, \quad z_1 = \frac{FA}{DH}\cdot\frac{1}{z}$$

transforms the curves $\rho_1 = C$, $\rho_2 = C$ into the curves $\rho_3 = C$, $\rho_4 = C$. These curves form two distinct pairs of fam lies of the same species.

We shall now discuss the special case where $\Delta = 0$. *The two pairs of families are identical.* We shall prove that in this case *the surface has four double points;* that is, the surface is of the fourth class and is *a tetrahedral symmetrical surface.*

There are 15 right lines on the surface (17) in the finite part of space, the remaining 12 being the three edges of the tetrahedron in the plane at infinity, each counted 4 times, since they join the double points of the surfaces. Of these 15 lines 12 consist of 6 double lines. We shall prove that when $\Delta = 0$ *these 6 pairs become three quadruple lines joining a fourth double point to the three already existing.*

Putting $z = k_1$ in (18), the resulting conic will represent two straight lines, if the determinant

$$\begin{vmatrix} 0 & k_1 H + F & k_1 E + B \\ k_1 H + F & 0 & k_1 G + C \\ k_1 E + B & k_1 G + C & 2(k_1 D + A) \end{vmatrix} = 0,$$

which gives us the following values for k_1:

$$k_1 = -\frac{F}{H}, \; (GE - HD)k_1^2 + (CE + BG - AH - FD)k_1 + BC - AF = 0. \quad (30)$$

In the same way, putting $y = k_2$, and $x = k_3$ in succession we find that k_2 and k_3 are determined by the following equations:

$$k_2 = -\frac{E}{H}, \; (GF - CH)k_2^2 + (BG + DF - CE - AH)k_2 + BD - EH = 0, \quad (31)$$

$$k_3 = -\frac{G}{H} (EF - BH)k_3^2 + (CE + FD - BG - AH)k_3 + CD - AG = 0. \quad (32)$$

If the roots of the second of equations (30) are equal, we have

$$\Delta_1 = (EC + BG - AH - FD)^2 - 4(BC - AF)(GE - HD) = 0,$$

from which it follows that also

$$\Delta_2 = (BG + DF - CE - AH)^2 - 4(BD - EA)(GF - CH) = 0.$$

In fact, we have

$$C^2E^2 + B^2G^2 + A^2H^2 + F^2D^2 + 2ECBG - 2ECAH - 2ECFD - 2BGAH$$
$$- 2BGFD + 2AHFD = 4(BGCE + AFDH - AFGE - BCHD);$$

adding to each side of this equation $4CEAH - 4CEBG + 4GBDF - 4DFAH$ we have

$$(GB + DF - CE - AH)^2 = 4(BD - EA)(GF - CH).$$

Hence $\Delta_2 = 0$ whenever $\Delta_1 = 0$; in the same way it may be proved that $\Delta_3 = 0$. But $\Delta_1 = \Delta$, since $GE - HD = \frac{BC}{EG}(CB - AF)$, so that *when* $\Delta \equiv 0$ *the three line-pairs degenerate into three double lines which all meet in a fourth double point whose coordinates are*

$$\begin{aligned} x_1 &= \frac{CE + FD - BG - AH}{2(BH - EF)} = k_3, \\ y &= \frac{BG + DF - CE - AH}{2(CH - GF)} = k_2, \\ z &= \frac{EC + BG - AH - FD}{2(DH - EG)} = k_1. \end{aligned} \quad (33)$$

We have thus proved the following

THEOREM V. *A cubic surface of the form*

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \tag{9}$$

whose coefficients satisfy the relation

$$(GB + EC - FD - AH)^2 = 4(BC - AF)(GE - HD),$$

has 4 *double points; three of these are situated at the three vertices of the tetrahedron of reference in the plane at infinity, while the fourth is the point* $x = k_3$, $y = k_2$, $z = k_1$.

If the surface has a center of involution (*i. e.*, $AEGF = BCDH$), the coordinates of the fourth double point are:

$$x = \left(\frac{CD}{EF}\right)^{\frac{1}{2}}, \quad y = \left(\frac{BD}{GF}\right)^{\frac{1}{2}}, \quad z = -\left(\frac{BC}{EG}\right)^{\frac{1}{2}}. \tag{34}$$

Transforming the origin to this point the surface is thrown into the well-known form

$$Pxy + Qxz + Ryz + Sxyz = 0, \tag{35}$$

a surface belonging to the type known as a *tetrahedral symmetrical* surface.

It is also evident geometrically that if one line-pair of (9) degenerates into a double line the surface must have a fourth double point; in fact, each one is a double line, since it passes through one of the double points at infinity; hence, when they coincide we have a quadruple line, which means that there must be a fourth double point. The surface is therefore reducible to the form (35).

The most general form of a tetrahedral symmetrical surface is

$$Ax^m + By^m + Cz^m + D = 0, \tag{36}$$

which, when $m = -1$ reduces to (35). We may therefore generalize our result as follows:

The transformation

$$x = x_1^m, \quad y = y_1^m, \quad z = z_1^m \tag{37}$$

transforms the surface (9) into the form

$$A + Bx^m + Cy^m + Dz^m + Ex^m z^m + Fx^m y^m + Gy^m z^m + Hx^m y^m z^m = 0, \tag{38}$$

which has the same characteristic properties as (9) viz.:

1°. *It contains a fourfold family of curves* $\rho_1, \rho_2, \rho_3, \rho_4$ *such that the pair* ρ_1 *and* ρ_2 *belong to the same species, and likewise* ρ_3 *and* ρ_4.

2°. *The* ∞^7 *surfaces* (38) *remain invariant for the transformation*

$$x=\lambda x_1, \quad y=\mu y_1, \quad z=\nu z_1.$$

3°. *Any one of the surfaces* (38) *remains invariant for the involutory transformation*

$$x^m=\frac{AG}{BH}\cdot\frac{1}{x_1^m}, \quad y^m=\frac{EA}{CH}\cdot\frac{1}{y_1^m}, \quad z^m=\frac{FA}{DH}\cdot\frac{1}{z_1^m}. \tag{37'}$$

4°. *By this transformation the curves* $\rho_1=C$, $\rho_2=C$ *are transformed into* $\rho_3=C$, $\rho_4=C$.

5°. *It is always possible to find a parametric representation of the surface* (38) *of the form*

$$x=\sqrt[m]{\frac{(\rho_1-\beta_1)(\rho_2-\beta_1)}{(\gamma_1\rho_1-\delta_1)(\gamma_1\rho_2-\delta_1)}}, \quad y=\sqrt[m]{\frac{(\rho_1-\beta_2)(\rho_2-\beta_2)}{(\gamma_2\rho_1-\delta_2)(\gamma_2\rho_1-\delta_2)}}, \quad z=\sqrt[m]{\rho_1\rho_2}, \tag{38'}$$

where β_1^2, β_2^2, γ_1^2, γ_2^2, δ_1^2, δ_2^2 *are roots of certain quadratic equations. A second mode of representation is obtained by taking for* $\beta_1^2, \ldots, \delta_2^2$ *their respective conjugate values, so that the same surface* (38) *is represented by the equations*

$$x=\sqrt[m]{\frac{(\rho_3-\beta_1')(\rho_4-\beta_1')}{(\gamma_1'\rho_3-\delta_1')(\gamma_1'\rho_4-\delta_1')}}, \quad y=\sqrt[m]{\frac{(\rho_3-\beta_2')(\rho_4-\beta_2')}{(\gamma_2'\rho_3-\delta_2')(\gamma_2'\rho_4-\delta_2')}}, \quad z=\frac{1}{\sqrt[m]{\rho_3\rho_4}}. \tag{38''}$$

$\Delta=0$ *is the condition that these surfaces shall be tetrahedral symmetrical.*

It should be noticed that on all the surfaces (19) and (38) there exist two special asymptotic curves $\rho_1=\rho_2$ and $\rho_3=\rho_4$. *When* $\Delta=0$ *the asymtotic lines may be determined by quadratures according to a theorem proved by Lie.** Moreover, in this case there can be drawn through any point of the surface ∞^1 curves of the same species, so that, instead of a fourfold family, we obtain ∞^1 families of curves of the same species. In fact, if we transform the origin to the fourth double point the surface takes the form

$$Pxy+Qxz+Ryz+Sxyz=0, \tag{39}$$

to which corresponds in the (X, Y, Z) space a translation-surface

$$Pe^{X+Y}+Qe^{X+Z}+Re^{Y+Z}+Se^{X+Y+Z}=0,$$

or, putting $x=-x$, $y=-y$, $z=-z$,

$$Pe^{Z}+Qe^{Y}+Re^{X}+S=0, \tag{40}$$

* Lie-Scheffers, "Geometrie der Berührungstransformationen," p. 341.

a translation-surface connected with a degenerate quartic consisting of two intersecting conics; this translation-surface has, moreover, ∞ families of translation-curves, as was proved by Lie.* Since, therefore, (40) can be generated in ∞^1 different ways, the surface (39) has ∞^1 families of curves of the same species. We thus see that *the families of surfaces* (9) *and* (38) *are, from our standpoint, the most natural generalization of the tetrahedral symmetrical surfaces.*

III.

We proved above that $\Delta = 0$ is the condition that the surface (19) shall reduce to the form (39). It is well known that the dualistic of this surface is the so-called Steiner's surface of the fourth order and third class, viz.:

$$\sqrt{Ru} + \sqrt{Qv} + \sqrt{Pw} + \sqrt{S} = 0. \tag{39'}$$

If, therefore, we form the dualistic of (9), we sha l obtain the generalization of Steiner's surface. This new surface will be of the sixth order.

In order to find its equation we proceed as follows: We transform the origin to the point $\frac{-G}{H}, \frac{-E}{H}, \frac{-F}{H}$, so that the surface takes the form

$$\begin{aligned} A - \frac{GB}{H} - \frac{EC}{H} - \frac{FD}{H} + \frac{2EGF}{H^2} + \frac{BH - FE}{H}x + \frac{CH - GF}{H}y \\ + \frac{DH - EG}{H}z + Hxyz = 0, \end{aligned} \tag{40}$$

which we write

$$A_1 + B_1 x + C_1 y + D_1 z + Hxyz = 0.$$

The tangential coordinates are now

$$\begin{aligned} \rho u &= B_1 + Hyz, \\ \rho v &= C_1 + Hxz, \\ \rho w &= D_1 + Hxy, \\ \rho &= 3A_1 + 2B_1 x + 2C_1 y + 2D_1 z = A_1 - 2Hxyz, \end{aligned} \tag{41}$$

from which we have $Hxyz = \frac{A_1 - \rho}{2}$. From the first three equations (41) we get

$$\frac{H}{4}(A_1 - \rho)^2 = (u\rho - B_1)(v\rho - C_1)(w\rho - D_1)\,; \tag{42}$$

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* See "Geometrie der Berührungstransformationen," p. 407.

from the equations (41) and

$$Hxyz = -(A_1 + B_1x + C_1y + D_1z) = \frac{A_1 - \rho}{2},$$

we get, after some reductions,

$$\begin{aligned} B_1(v\rho - C_1)(w\rho - D_1) + C_1(u\rho - B_1)(w\rho - D_1) + D_1(u\rho - B_1)(v\rho - C_1) \\ = \frac{H}{4}(A_1 - \rho)^2 - \frac{A_1H}{2}(A_1 - \rho). \end{aligned} \tag{43}$$

Equations (41) and (43) being respectively a cubic and a quadratic in ρ, we may eliminate ρ by Sylvester's dialytic method. We write the equations

$$\begin{aligned} a\rho^3 + b\rho^2 + c\rho + d = 0, \\ p\rho^2 + q\rho + r = 0, \end{aligned} \tag{44}$$

where the coefficients have the following values:

$$\begin{aligned} &a = uvw,\ b = -\left(B_1vw + C_1uw + D_1uv + \frac{H}{4}\right), \\ &c = C_1D_1u + B_1D_1v + B_1C_1w + \frac{A_1H}{2}, \\ &d = -\left(B_1C_1D_1 + \frac{A_1^2H}{4}\right), \\ &p = B_1vw + C_1uw + D_1uv + \frac{H}{4}, \\ &q = -2\left(C_1B_1w + D_1B_1v + C_1D_1u + \frac{A_1H}{2}\right), \\ &r = 3\left(B_1C_1D_1 + \frac{A_1^2H}{4}\right), \end{aligned} \tag{45}$$

which show that the following relations exist between the coefficients

$$p = -b, \qquad q = -2c, \qquad r = -3d.$$

Eliminating ρ from (44), we have

$$\begin{vmatrix} a & b & c & d & 0 \\ 0 & a & b & c & d \\ -b & -2c & -3d & 0 & 0 \\ 0 & -b & -2c & -3d & 0 \\ 0 & 0 & -b & -2c & -3d \end{vmatrix} = 0, \tag{46}$$

which, developed, gives us

$$27a^2d^2 - 18abcd + 4ac^3 - b^2c^2 + 4b^3d = 0,$$

a surface of the sixth degree in u, v, w.

If $d = 0$, the surface (46) reduces to

$$(4ac - b^2)c^2 = 0.$$

$c^2 = 0$ represents a double point, while the factor

$$4ac - b^2 = 0$$

represents *Steiner's surface.* In fact, if we write

$$d = 4(BH - FE)(CH - GF)(DH - EG) + [(AH - GB - EC - FD)H + 2EGF]^2,$$

it will easily be seen to vanish whenever $\Delta = 0$. A little calculation will show that $d = H^2\Delta$. We get for Steiner's surface

$$4uvw\left(C_1D_1u + B_1D_1v + B_1C_1w + \frac{A_1H}{2}\right) - \left(B_1uv + C_1uw + B_1vw + \frac{H}{4}\right)^2 = 0, \tag{47}$$

which is somewhat different in form from the one usually given, owing to the different method of obtaining the equation. (For the regular form see Salmon, "Geometry of Three Dimensions," p. 491, note.)

Returning to the surface (46), we shall prove that it has *one triple point, three double edges, and a cuspidal curve of order* 6.

We shall use Salmon's equations connecting the singularities of a surface, viz.:*

$$\begin{aligned} a'(n'-2) &= k' + \rho' + 2\sigma', \\ b'(n'-2) &= \rho' + 2\beta' + 3\gamma' + 3t', \\ c'(n'-2) &= 2\sigma' + 4\beta' + \gamma', \\ a' + 2b' + 3c' &= n'(n'-1), \end{aligned} \tag{48}$$

where the letters have the meaning explained on pp. 580 and 581 of Salmon's treatise.

The points σ' are the intersections of an arbitrary plane with the curve UH, where U is the cubic surface and H the Hessian. Ordinarily there are 12 of these, but if we form the Hessian of the surface (19), we find that it intersects the cubic in a curve of the sixth order; hence $\sigma' = 6$. k' is the number of cuspidal edges on the tangent cone proper, and equals 9. a' is the class of a plane section of the cubic, hence $a' = 6$. γ' is zero, as always in the case of the dualistic of a cubic surface. Substituting these values in the above equations

*Salmon, "Analytical Geometry of Three Dimensions," third ed., p. 580.

and putting $n' = 6$, we find $\rho' = 3$, $\beta' = 3$, $b' = 3$, $t' = 1$ and $c' = 6$, c' being the order of the cuspidal cubic, b' that of the double line, and t' the number of triple points. *When* $\Delta = 0$ *the cuspidal edge vanishes*; in fact, if we form the Hessian of a cubic surface with four double points, we find that it is tangent to the surface and does not intersect it. Hence $c' = 0$; the triple point and the double edges remain: the surface (19) has degenerated into a Steiner surface.

THEOREM VI. *The curves $\rho_1, \rho_2, \rho_3, \rho_4$ constitute a fourfold family of cubics on* (46). *These arrange themselves into two pairs which, when* $\Delta = 0$, *reduce to a single pair.*

Proof. The tangential coordinates of (18) are

$$\begin{aligned}\rho u &= B + Ez + Fy + Hyz,\\ \rho v &= C + Fx + Gz + Hxz,\\ \rho w &= D + Ex + Gy + Hxy,\\ \rho &= 2A + Bx + Cy + Dz - Hxyz.\end{aligned}$$

Introducing in these equations the values of x, y, z from (17), we obtain the parametric representation of the dualistic surfaces and it is easily seen that the curves ρ_1 and ρ_2 are still twisted cubics; the same will also hold if we substitute the values of x, y and z obtained from equations (17″). When $\Delta = 0$, these two representations are identical; it should be noticed that these curves are not of the same species.*

The relation $AEGF = BCDH$ which we assumed to hold is a purely metrical one. Given any surface of the form

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0, \tag{49}$$

we may by a simple translation throw it into a similar form and such that the new coefficients satisfy this relation; there exist ∞^2 such translations, since the coordinates of the new origin must satisfy one relation.

To find this we put $x = x_1 + \xi$, $y = y_1 + \eta$, $z = z_1 + \zeta$ in (49). The new equation may be written

$$A_1 + B_1x_1 + C_1y_1 + D_1z_1 + E_1x_1z_1 + F_1x_1y_1 + G_1y_1z_1 + H_1x_1y_1z_1, \tag{49}$$

where the coefficients have the following values:

$$\begin{aligned}&A_1 = A + B\xi + C\eta + D\zeta + E\xi\zeta + F\xi\eta + G\eta\zeta + H\xi\eta\zeta,\\ &B_1 = B + E\zeta + F\eta + H\eta\zeta, \quad C_1 = C + F\xi + G\zeta + H\xi\zeta,\\ &D = D + E\xi + G\eta + H\xi\eta, \quad E_1 = E + H\eta,\\ &F_1 = F + H\zeta, \quad G_1 = G + H\xi, \quad H_1 = H.\end{aligned}$$

* On Steiner's surface the curves of the same species are quartic curves according to Lie. See Lie-Scheffers, "Geometrie der Berührungstransformationen," p. 333.

Forming the identity $A_1E_1G_1F_1 = B_1C_1D_1H_1$, we find that ξ, η, ζ must be a point on the cubic surface

$$\begin{aligned} T = & AEGF - BCDH + (BEFG + AEFH - BDHF - ECBH)\xi \\ & + (EFGC + AFGH - CDFH - BGCH)\eta \\ & + (DEFG + AGEH - ECDH - BDGH)\zeta \\ & + (AFH^2 + EGF^2 - HDF^2 - CBH^2)\xi\eta \\ & + (AEH^2 + FGE^2 - DBH^2 - CHE^2)\xi\zeta \\ & + (EFG^2 + AGH^2 - BHG^2 - CDH^2)\eta\zeta \\ & + (AH^3 + 4HEGF + DFH^2 + CEH^2 + GBH^2)\xi\eta\zeta = 0. \end{aligned}$$

If then ξ, η, ζ lies on this surface and if the values of these coordinates do not cause any of the coefficients $A_1, B_1, \ldots, H_1$ to vanish, the resulting transform of (49) *will have a center of involution.* By proceeding with (49′) as we did with (18), the curves ρ_1, ρ_2, ρ_3 and ρ_4 may be obtained by solving a quadratic equation. Now any cubic surface having three double points may by a projective transformation be put in the form (49); hence the

THEOREM VII. *A cubic surface having three double points of which none are biplanar, may by a translation be put in the form*

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyz = 0,$$

where the coefficients satisfy the identical relation

$$AEGF = BCDH.$$

The center of involution, which is

$$\frac{\sqrt{AG}}{\sqrt{BH}}, \quad \frac{\sqrt{EA}}{\sqrt{CH}}, \quad \frac{\sqrt{FA}}{\sqrt{DH}},$$

may be chosen in ∞^2 ways. The surface having been thrown into the form (18), *the fourfold family of curves of the same species may be determined by solving a quadratic equation.*

The following example will show how the theory works. Let the surface be

$$3(xyz - 1) + 2(x - yz) + 2(y - xz) + 2(z - xy) = 0. \tag{50}$$

We have here,

$$AB = -6, \ ED = -4, \ GB = -4, \ EC = -4, \ FD = -4, \ AH = -9,$$

and the equation (25), p. 9, becomes

$$\beta_2^4 - \frac{15}{4}\beta_2^2 + \frac{9}{4} = 0.$$

Solving, we find

$$\beta_2^2 = 3, \quad \beta_2'^2 = \frac{3}{4}.$$

Using the first value, we find, by substituting in equations (21) and (23),

$$\beta_1^2 = \frac{3}{4}, \quad \gamma_1^2 = \frac{3}{4}, \quad \delta_1^2 = \frac{1}{4},$$
$$\beta_2^2 = 3, \quad \gamma_2^2 = 3, \quad \delta_2^2 = 4.$$

Since $\frac{A}{D}$ is negative β_1 and β_2 must have like signs; thus, if we put $\beta_1 = \frac{1}{2}\sqrt{3}$, we have $\beta_2 = \sqrt{3}$. $\frac{B}{E}$ being also negative and β_2 positive, it follows from the second of equations (21) that δ_1 and γ_1 must have like signs. We put $\gamma_1 = \frac{\sqrt{3}}{2}$, and $\delta_1 = \frac{1}{2}$. γ_2 and δ_2 must also have like signs; so we put $\gamma_2 = \sqrt{3}$ and $\delta_2 = 2$. The parametric representation of the surface (50) is therefore

$$x = \frac{(\rho_1 - \frac{1}{2}\sqrt{3})(\rho_2 - \frac{1}{2}\sqrt{3})}{(\frac{1}{2}\sqrt{3}\rho_1 - \frac{1}{2})(\frac{1}{2}\sqrt{3}\rho_2 - \frac{1}{2})}, \quad y = \frac{(\rho_1 - \sqrt{3})(\rho_2 - \sqrt{3})}{(\sqrt{3}\rho_1 - 2)(\sqrt{3}\rho_2 - 2)}, \quad z = \rho_1\rho_2. \quad (50')$$

Suppose now we take the second value $\beta_1'^2 = \frac{3}{4}$; we find

$$\beta_1'^2 = 3, \quad \gamma_1'^2 = 3, \quad \delta_1'^2 = 4,$$
$$\beta_2'^2 = \tfrac{3}{4}, \quad \gamma_2'^2 = \tfrac{3}{4}, \quad \delta_2'^2 = \tfrac{1}{4},$$

Following the same rules for signs, we get the second representation

$$x = \frac{(\rho_3 - \sqrt{3})(\rho_4 - \sqrt{3})}{(\sqrt{3}\rho_3 - 2)(\sqrt{3}\rho_4 - 2)}, \quad y = \frac{(\rho_3 - \frac{1}{2}\sqrt{3})(\rho_4 - \frac{1}{2}\sqrt{3})}{(\frac{1}{2}\sqrt{3}\rho_3 - \frac{1}{2})(\frac{1}{2}\sqrt{3}\rho_4 - \frac{1}{2})}, \quad z = \rho_3\rho_4, \quad (50'')$$

which is identical with

$$x = \frac{(\frac{1}{2}\sqrt{3}\rho_3' - \frac{1}{2})(\frac{1}{2}\sqrt{3}\rho_4' - \frac{1}{2})}{\left(\rho_3' - \frac{\sqrt{3}}{2}\right)(\rho_4' - \frac{1}{2}\sqrt{3})}, \quad y = \frac{(\sqrt{3}\rho_3' - 2)(\sqrt{3}\rho_4' - 2)}{(\rho_3' - \sqrt{3})(\rho_4' - \sqrt{3})}, \quad z = \frac{1}{\rho_3'\rho_4'},$$

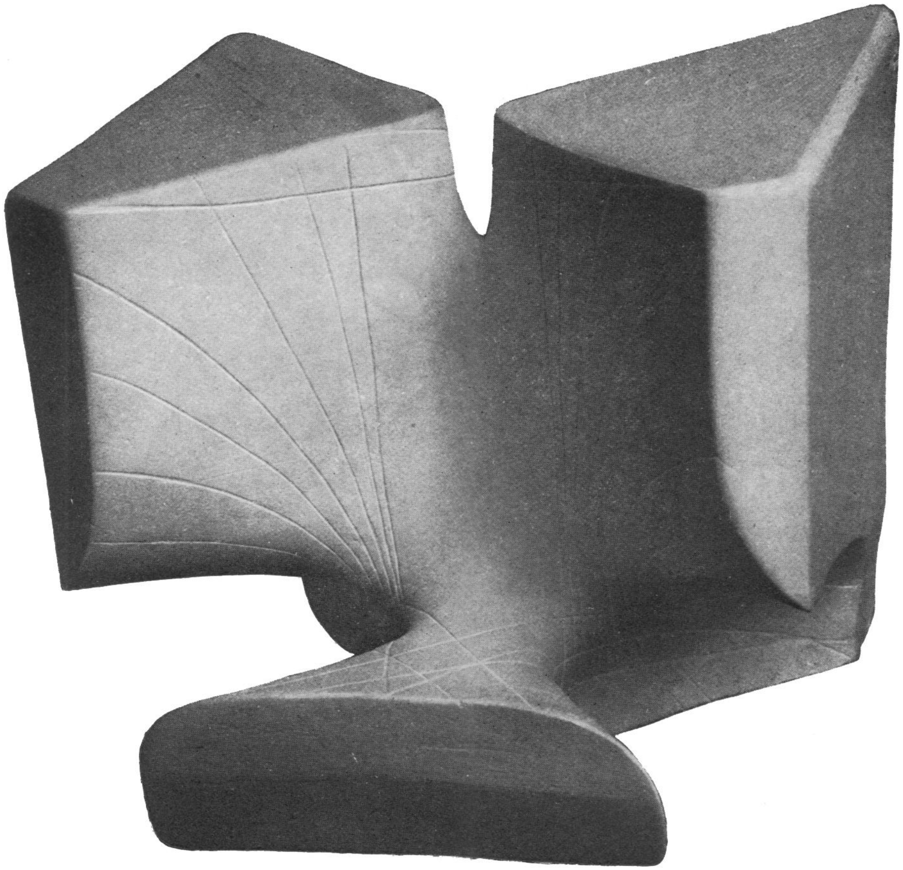
if we put $\frac{1}{\rho_3'} = \rho_3$ and $\frac{1}{\rho_4'} = \rho_4$. This surface has been modeled. (See plate.)

For the purpose of modeling it was found convenient to put $z = \frac{z'}{3}$ in (50), so that the equation of the surface becomes

$$3xyz - 2yz - 2xz - 6xy + 6x + 6y + 2z - 9 = 0,$$

which has for parametric representation

$$x = \frac{(\rho_1 - \frac{1}{2}\sqrt{3})(\rho_2 - \frac{1}{2}\sqrt{3})}{(\frac{1}{2}\sqrt{3}\rho_1 - \frac{1}{2})(\frac{1}{2}\sqrt{3}\rho_2 - \frac{1}{2})}, \quad y = \frac{(\rho_1 - \sqrt{3})(\rho_2 - \sqrt{3})}{(\sqrt{3}\rho_1 - 2)(\sqrt{3}\rho_2 - 2)}, \quad z = 3\rho_1\rho_2;$$



putting $\sqrt{3}\rho_1 = \rho_1'$, $\sqrt{3}\rho_2 = \rho_2'$, this may be written

$$x = \frac{4}{3} \cdot \frac{(\rho_1 - 3/2)(\rho_2 - 3/2)}{(\rho_1 - 1)(\rho_2 - 1)}, \quad y = \frac{1}{3} \cdot \frac{(\rho_1 - 3)(\rho_2 - 3)}{(\rho_1 - 2)(\rho_2 - 2)}, \quad z_1 = \rho_1\rho_2.$$

The curves $\rho_1, \rho_2, \rho_3, \rho_4$ have been shown on the model.

IV.

It remains to consider the case where the unicursal quartic, which was the starting-point of the preceding theory, degenerates into a unicursal cubic and a straight line.*

Let the cubic have a double point. By means of a projective transformation it may be thrown into the form $y(1 - x^2) = x^3$, while the straight line may be written $y = mx + b$. The quartic now is

$$F(xy) = [y(1 - x^2) - x^3][y - mx - b] = 0, \tag{51}$$

from which we obtain

$$\begin{aligned} F'_{y_1} &= \quad x_1^3 - (mx_1 + b)(1 - x_1^2) = \quad (1 + m)x_1^3 + bx_1^2 - mx_1 - b, \\ F'_{y_2} &= -x_2^3 + (mx_2 + b)(1 - x_2^2) = -[(1 + m)x_2^3 + bx_2^2 - mx_2 - b]. \end{aligned} \tag{52}$$

Forming the Abelian integrals of the first kind, according to Lie's theorem,† we have the translation-surface

$$\begin{aligned} X &= \int \frac{x_1 dx_1}{(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{x_2 dx_2}{(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)}, \\ Y &= \int \frac{x_1^3 dx_1}{(1 - x_1^2)(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{x_2^3 dx_2}{(1 - x_2^2)(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)}, \\ Z &= \int \frac{dx_1}{(x_1 - \alpha)(x_1 - \beta)(x_1 - \gamma)} - \int \frac{dx_2}{(x_2 - \alpha)(x_2 - \beta)(x_2 - \gamma)}, \end{aligned} \tag{53}$$

where α, β, γ are the roots of the cubic equation

$$(1 + m)x^3 + bx^2 - mx - b = 0;$$

we shall assume these roots real and different, that is to say, the straight line

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*The theory of translation-surfaces connected with a cubic and a straight line has been treated by Georg Wiegner in his thesis: *Ueber eine besondere Classe von Translationsflächen.* Inaugural dissertation. Leipzig, (1893). See also my paper in AM. JOUR. OF MATH., Vol. XXIX, p. 370. Wiegner does not treat the case of unicursal cubics separately; in fact, the rôle that unicursality plays in this theory was not known to him, although very important, as is seen from a theorem proved by me in a paper published in AM. JOUR. OF MATH., Vol. XXX, p. 179.

†See Lie-Scheffers, "Theorie der Berührungstransformationen," p. 411. A complete statement of the theorem is given in my paper, AM. JOUR. OF MATH., Vol. XXX, p. 171.

cuts the cubic in three distinct points. Transforming (53) by means of a linear projective transformation

$$X = Z', \quad Z = X', \quad mX - Y + bZ = Y',$$

we get a simpler form, viz.:

$$X' = \int \frac{dx_1}{(x_1-\alpha)(x_1-\beta)(x_1-\gamma)} - \int \frac{dx_2}{(x_2-\alpha)(x_2-\beta)(x_2-\gamma)},$$

$$Y' = \int \frac{dx_1}{1-x_1^2},$$

$$Z' = \int \frac{x_1 dx_1}{(x_1-\alpha)(x_1-\beta)(x_1-\gamma)} - \int \frac{x_2 dx_2}{(x_2-\alpha)(x_2-\beta)(x_2-\gamma)}.$$

Integrating and putting $X'' = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)X'$, $Y'' = 2Y'$ and $Z'' = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)Z$, we obtain (dropping primes),

$$e^X = \frac{(x_1-\alpha)^{\gamma-\beta}(x_1-\beta)^{\alpha-\gamma}(x_1-\gamma)^{\beta-\alpha}}{(x_2-\alpha)^{\gamma-\beta}(x_2-\beta)^{\alpha-\gamma}(x_2-\gamma)^{\beta-\alpha}},$$

$$e^Y = \frac{1+x_1}{1-x_1}, \tag{54}$$

$$e^Z = \frac{(x_1-\alpha)^{\alpha(\gamma-\beta)}(x_1-\beta)^{\beta(\alpha-\gamma)}(x_1-\gamma)^{\gamma(\beta-\alpha)}}{(x_2-\alpha)^{\alpha(\gamma-\beta)}(x_2-\beta)^{\beta(\alpha-\gamma)}(x_2-\gamma)^{\gamma(\beta-\alpha)}},$$

By using a proper linear transformation of the coordinates X, Y and Z, which is not difficult to find, we may bring the surface (54) into the equivalent form

$$e^X = \frac{(x_1-\beta)(x_2-\gamma)}{(x_1-\gamma)(x_2-\beta)}, \quad e^Y = \frac{1+x_1}{1-x_1}, \quad e^Z = \frac{(x_1-\alpha)(x_2-\gamma)}{(x_1-\gamma)(x_2-\alpha)}. \tag{55}$$

Eliminating x_1 and x_2, we have the surface

$$\begin{aligned}(\alpha-\beta)(1-\gamma)e^{X+Y+Z} + (1+\gamma)(\beta-\alpha)e^{X+Z} + (1-\beta)(\gamma-\alpha)e^{Y+Z} \\ + (1-\alpha)(\beta-\gamma)e^{X+Y} + (1+\beta)(\alpha-\gamma)e^Z + (1+\alpha)(\gamma-\beta)e^X = 0.\end{aligned} \tag{55'}$$

Since α, β and γ depend on the two parameters m and b, we may say:

To a unicursal cubic and a straight line in the plane at infinity there correspond ∞^2 *types of translation-surfaces of the form* (55').

It now remains to find the second pair of translation-curves on (55'). Since now we can not fall back on the principle of symmetry, which holds only in the case of irreducible quartics, we have to begin *ab initio*. We write the quartic as before,

$$F(xy) = [y(1-x^2) - x^3][y - mx - b] = 0,$$

but now we choose the two intersections of the variable line with $F(xy)=0$ on the cubic, since before we took one point on the cubic and one on the straight line $y=mx+b$. We have

$$F'_{y_3}=x_3^3-(mx_3+b)(1-x_3^2)=(1+m)x_3^3+bx_3^2-mx_3-b$$
$$F'_{y_4}=x_4^3-(mx_4+b)(1-x_4^2)=(1+m)x_4^3+bx_4^2-mx_4-b.$$

Forming the Abelian integrals as before, we get

$$\begin{aligned}
X&=\int\frac{x_3dx_3}{(x_3-\alpha)(x_3-\beta)(x_3-\gamma)}+\int\frac{x_4dx_4}{(x_4-\alpha)(x_4-\beta)(x_4-\gamma)},\\
Y&=\int\frac{x_3^3dx_3}{(1-x_3^2)(x_3-\alpha)(x_3-\beta)(x_3-\gamma)}+\int\frac{x_4^3dx_4}{(1-x_4^2)(x_4-\alpha)(x_4-\beta)(x_4-\gamma)},\\
Z&=\int\frac{dx_3}{(x_3-\alpha)(x_3-\beta)(x_3-\gamma)}+\int\frac{dx_4}{(x_4-\alpha)(x_4-\beta)(x_4-\gamma)},
\end{aligned}\tag{56}$$

which we shall transform, putting

$$X'=X,\quad Y'=-mX+Y-bZ,\quad Z'=Z,$$

so that (56) reduces to

$$\begin{aligned}
X&=\int\frac{x_3dx_3}{(x_3-\alpha)(x_3-\beta)(x_3-\gamma)}+\int\frac{x_4dx_4}{(x_4-\alpha)(x_4-\beta)(x_4-\gamma)},\\
Y&=\int\frac{dx_3}{1-x_3^2}+\int\frac{dx_4}{1-x_4^2},\\
Z&=\int\frac{dx_3}{(x_3-\alpha)(x_3-\beta)(x_3-\gamma)}+\int\frac{dx_4}{(x_4-\alpha)(x_4-\beta)(x_4-\gamma)}.
\end{aligned}\tag{56'}$$

Integrating and transforming as before we have

$$\begin{aligned}
e^X&=[(x_3-\alpha)(x_4-\alpha)]^{\alpha(\gamma-\beta)}[(x_3-\beta)(x_4-\beta)]^{\beta(\alpha-\gamma)}[(x_3-\gamma)(x_4-\gamma)]^{\gamma(\beta-\alpha)},\\
e^Y&=\frac{1+x_3}{1-x_3}\cdot\frac{1+x_4}{1-x_4},\\
e^Z&=[(x_3-\alpha)(x_4-\alpha)]^{\gamma-\beta}[(x_3-\beta)(x_4-\beta)]^{\alpha-\gamma}[(x_3-\gamma)(x_4-\gamma)]^{\beta-\alpha}.
\end{aligned}\tag{56''}$$

Performing certain linear transformations on the variables X and Z and putting $Y=-Y$, we reduce (56″) to the form

$$e^X=\frac{(x_3-\gamma)(x_4-\gamma)}{(x_3-\alpha)(x_4-\alpha)},\quad e^Y=\frac{1-x_3}{1+x_3}\cdot\frac{1-x_4}{1+x_4},\quad e^Z=\frac{(x_3-\gamma)(x_4-\gamma)}{(x_3-\beta)(x_4-\beta)}.\tag{57}$$

Eliminating x_3 and x_4, we obtain the surface

$$\begin{aligned}
&(\beta-\alpha)(\alpha+1)(\beta+1)e^{X+Y+Z}+(\gamma-\beta)(\beta+1)(\gamma+1)e^{X+Y}\\
&+(\alpha-\beta)(\alpha+1)(\gamma+1)e^{Y+Z}\quad+(\alpha-\beta)(\alpha-1)(\beta-1)e^{X+Z}\\
&+(\beta-\gamma)(\beta-1)(\gamma-1)e^{X}\quad+(\gamma-\alpha)(\alpha-1)(\gamma-1)e^{Z}=0.
\end{aligned}\tag{57'}$$

If now we translate this surface to a point ξ, η, ζ, it must be possible to determine these coordinates in such a way that the surface becomes identical with (55′). Putting therefore $X = X' + \xi$, $Y = Y' + \eta$, $Z = Z' + \zeta$ in (57′) and putting the new coefficients equal to const. $\times$ the corresponding coefficients of (55′), we find, after easy calculations,

$$e^{\xi} = \frac{\gamma^2 - 1}{\beta^2 - 1}, \quad e^{\eta} = \frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\gamma)(1-\alpha)(1-\beta)}, \quad e^{\zeta} = \frac{\alpha^2 - 1}{\gamma^2 - 1}.$$

The second mode of representation is therefore

$$\begin{gathered} e^X = \frac{\beta^2 - 1}{\gamma^2 - 1}\frac{(x_3-\gamma)(x_4-\gamma)}{(x_3-\beta)(x_4-\beta)}, \quad e^Y = \frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\alpha)(1-\beta)(1-\gamma)}\frac{(1-x_3)(1-x_4)}{(1+x_3)(1+x_4)}, \\ e^Z = \frac{\alpha^2-1}{\gamma^2-1}\frac{(x_3-\gamma)(x_4-\gamma)}{(x_3-\alpha)(x_4-\alpha)}. \end{gathered} \tag{55'}$$

Hence the

THEOREM VIII. *To a cubic with a double point and a straight line there correspond ∞^2 types of translation-surfaces that can be generated in four different ways. These surfaces are of the form*

$$Ae^{X+Y+Z} + Be^{X+Z} + Ce^{Y+Z} + De^{X+Y} + Ee^X + Fe^Z = 0. \tag{60}$$

The converse will follow from a theorem which we shall prove later. If we transform (60), putting $X = -X$, $Y = -Y$, $Z = -Z$, we obtain a somewhat simpler form

$$A + Be^Y + Ce^X + De^Z + Ee^{Y+Z} + Fe^{X+Y} = 0, \tag{60'}$$

which is represented parametrically by the two sets of equations

$$e^X = \frac{(x_1-\gamma)(x_2-\beta)}{(x_1-\beta)(x_2-\gamma)}, \quad e^Y = \frac{1-x_1}{1+x_2}, \quad e^Z = \frac{(x_1-\gamma)(x_2-\alpha)}{(x_1-\alpha)(x_2-\gamma)}. \tag{60''}$$

$$\begin{gathered} e^X = \frac{\gamma^2-1}{\beta^2-1}\cdot\frac{(x_3-\beta)(x_4-\beta)}{(x_3-\gamma)(x_4-\gamma)}, \quad e^Y = \frac{(1-\alpha)(1-\beta)(1-\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)}\cdot\frac{1+x_3}{1-x_3}\cdot\frac{1+x_4}{1-x_4}, \\ e^Z = \frac{\gamma^2-1}{\alpha^2-1}\frac{(x_3-\alpha)(x_4-\alpha)}{(x_3-\gamma)(x_4-\gamma)}. \end{gathered} \tag{60'''}$$

We shall now introduce the logarithmic transformation that we have used before, viz.:

$$e^X = x, \; e^Y = y, \; e^Z = z;$$

and we shall introduce the parameters $\rho_1, \rho_2, \rho_3, \rho_4$ instead of x_1, x_2, x_3, x_4 respectively. The surface (60) is thereby transformed into a cubic having three

double points, and passing through all four vertices of the tetrahedron and through the edge $x=0$, $z=0$. This surface is

$$Axyz+Bxz+Cyz+Dxy+Ex+Fz=0, \tag{61}$$

which contains 4 families of curves that group themselves in pairs, each pair belonging to the same species; one pair, $\rho_1=c$, $\rho_2=c$, are plane conics; while the second pair, $\rho_3=c$, $\rho_4=c$, are twisted cubics. It should be noticed that now *there is no involutory transformation* (4) *which will transform* ρ_1 *and* ρ_2 *into* ρ_3 *and* ρ_4, *since the corresponding translation-surface has no center of symmetry.* The parametric equations are

$$x=\frac{\rho_1-\beta}{\rho_1-\gamma}\cdot\frac{\rho_2-\gamma}{\rho_2-\beta},\quad y=\frac{1+\rho_1}{1-\rho_1},\quad z=\frac{\rho_1-\alpha}{\rho_1-\gamma}\cdot\frac{\rho_2-\gamma}{\rho_2-\alpha}, \tag{61'}$$

or,

$$\begin{gathered}x=\frac{\beta^2-1}{\gamma^2-1}\cdot\frac{(\rho_3-\gamma)(\rho_4-\gamma)}{(\rho_3-\beta)(\rho_4-\beta)},\quad y=\frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\alpha)(1-\beta)(1-\gamma)}\cdot\frac{1-\rho_3}{1+\rho_3}\cdot\frac{1-\rho_4}{1+\rho_4},\\ z=\frac{\alpha^2-1}{\gamma^2-1}\cdot\frac{(\rho_3-\gamma)(\rho_4-\gamma)}{(\rho_3-\alpha)(\rho_4-\alpha)}.\end{gathered} \tag{61''}$$

If we use the second form (60′), we obtain the surface

$$A+By+Cx+Dz+Eyz+Fxy=0, \tag{62}$$

which may be obtained from (61) by the involutory transformation

$$x_1=\frac{1}{x},\quad y_1=\frac{1}{y},\quad z_1=\frac{1}{z}.$$

This surface likewise has 4 sets of curves; *both pairs are conic sections, each pair being curves of the same species.* The parametric equations are

$$x=\frac{\rho_1-\gamma}{\rho_1-\beta}\cdot\frac{\rho_2-\beta}{\rho_2-\gamma},\quad y=\frac{1-\rho_1}{1+\rho_1},\quad z=\frac{\rho_1-\gamma}{\rho_1-\alpha}\cdot\frac{\rho_2-\alpha}{\rho_2-\gamma}, \tag{62'}$$

or

$$\begin{gathered}x=\frac{\gamma^2-1}{\beta^2-1}\cdot\frac{\rho_3-\beta}{\rho_3-\gamma}\cdot\frac{\rho_4-\beta}{\rho_4-\gamma},\quad y=\frac{(1-\alpha)(1-\beta)(1-\gamma)}{(1+\alpha)(1+\beta)(1+\gamma)}\cdot\frac{1+\rho_3}{1-\rho_3}\cdot\frac{1+\rho_4}{1-\rho_4},\\ z=\frac{\gamma^2-1}{\alpha^2-1}\cdot\frac{\rho_3-\alpha}{\rho_3-\gamma}\cdot\frac{\rho_4-\alpha}{\rho_4-\gamma}.\end{gathered} \tag{62''}$$

We shall now prove the following

THEOREM IX. *Given any cubic surface of the form*

$$Axyz+Bxz+Cyz+Dxy+Ex+Fz=0 \tag{63}$$

with non-vanishing coefficients, it is always possible to find a double parametric representation of the form (61′) *and* (61″).

Proof. Let the surface (63) be transformed by means of the transformation

$$x = \lambda x_1, \quad y = \mu y_1, \quad z = \nu z_1.$$

The new surface,

$$A\lambda\mu\nu x_1 y_1 z_1 + B\lambda\nu x_1 z_1 + C\mu\nu y_1 z_1 + D\lambda\mu x_1 y_1 + E\lambda x_1 + F\nu z_1 = 0, \tag{64}$$

will be identical with (61′) if we put

$$\begin{aligned} \lambda\mu\nu A &= (1-\gamma)(\alpha-\beta), & \lambda\nu B &= (1+\gamma)(\beta-\alpha), \\ \mu\nu C &= (1-\beta)(\gamma-\alpha), & \lambda\mu D &= (1-\alpha)(\beta-\gamma), \\ \lambda E &= (1+\alpha)(\gamma-\beta), & \nu F &= (1+\beta)(\alpha-\gamma). \end{aligned} \tag{65}$$

If these equations can be solved for $\lambda, \mu, \nu, \alpha, \beta, \gamma$, we shall have a parametric representation of (63) by putting

$$x = \lambda \cdot \frac{(\rho_1-\gamma)(\rho_2-\beta)}{(\rho_1-\beta)(\rho_2-\gamma)}, \quad y = \mu \frac{1-\rho_1}{1+\rho_2}, \quad z = \nu \frac{(\rho_1-\gamma)(\rho_2-\alpha)}{(\rho_1-\alpha)(\rho_2-\gamma)},$$

or else

$$x = \lambda \cdot \frac{\beta^2-1}{\gamma^2-1} \cdot \frac{(\rho_3-\gamma)(\rho_4-\gamma)}{(\rho_3-\beta)(\rho_4-\beta)}, \quad y = \mu \frac{(1+\alpha)(1+\beta)(1+\gamma)}{(1-\alpha)(1-\beta)(1-\gamma)} \frac{1-\rho_3}{1+\rho_3} \cdot \frac{1-\rho_4}{1+\rho_4},$$

$$z = \nu \cdot \frac{\alpha^2-1}{\gamma^2-1} \cdot \frac{(\rho_3-\gamma)(\rho_4-\gamma)}{(\rho_3-\alpha)(\rho_4-\alpha)}.$$

We get from (65)

$$\mu \frac{A}{B} = \frac{\gamma-1}{\gamma+1}, \quad \mu \frac{D}{E} = \frac{\alpha-1}{\alpha+1}, \quad \mu \frac{C}{F} = \frac{\beta-1}{\beta+1},$$

$$\gamma = \frac{B+\mu A}{B-\mu A}, \quad \beta = \frac{F+\mu C}{F-\mu C}, \quad \alpha = \frac{E+\mu D}{E-\mu D}.$$

Substituting the values of α, β and γ in the two equations

$$\frac{\lambda A}{C} = \frac{1-\gamma}{1-\beta} \cdot \frac{\alpha-\beta}{\gamma-\alpha}, \quad \frac{\gamma A}{D} = \frac{1-\gamma}{1-\alpha} \cdot \frac{\alpha-\beta}{\beta-\gamma},$$

obtained from (65), we get

$$\lambda = \frac{EC-DF}{BD-EA}, \quad \nu = \frac{EC-DF}{AF-BC},$$

while μ is a root of the cubic equation

$$(F-\mu C)(B-\mu A)(E-\mu D) = 4\mu \frac{(BD-EA)(AF-BC)}{EC-DF},$$

obtained from the first of equations (65). Q. E. D. To the above theorem may also be added

THEOREM X. *Given a quadric surface of the form*

$$A + By + Cx + Dz + Eyz + Fxy = 0 \tag{62}$$

with non-vanishing coefficients, it is always possible to find a double parametric representation (62′) *and* (63″). *The two pairs of families of conic sections belong each to the same species.*

The surfaces (62) *bear therefore the same relation to Lie's quadrics* (3) *that the cubic surfaces* (61) *do to the cubic surfaces* (9). There is an essential difference between the two categories: *There exists an involutory transformation that will transform the surfaces* (3) *and* (9) *into themselves, while the surfaces* (61) *and* (62) *do not admit of such a transformation, but are transformed into each other by the same transformation.*

These theorems are also true, as regards the property in question, for surfaces derived from (61) and (62) by means of the transformation

$$x = x_1^m, \quad y = y_1^m, \quad z = z_1^m;$$

that is, for the surfaces

$$Ax^m y^m z^m + Bx_1^m z_1^m + Cy_1^m z_1^m + Dx_1^m y_1^m + Ex_1^m + Fz_1^m = 0, \tag{66}$$

$$A + By_1^m + Cx_1^m + Dz_1^m + Ey_1^m z_1^m + Fx_1^m y_1^m = 0, \tag{67}$$

the parametric representations of which are easily deduced from equations (55) and (57). There exists a one-to-one correspondence between the surfaces (66) and (67) by virtue of the involutory transformation

$$x = \frac{1}{x_1}, \quad y = \frac{1}{y_1}, \quad z = \frac{1}{z_1}.$$

To the four families of curves of the same species on (66) correspond 4 families on (67); in fact, if in (66) we put $m = - m$ we obtain the surfaces (67).

If $\Delta = CE - DF = 0$, these surfaces also become tetrahedral symmetrical, in which case there exist ∞^1 families of curves of the same species.

CONCLUSION.

The preceding investigations have thus revealed an extensive class of surfaces on which there exist at least 4 families of curves of the same species. In case the determinant Δ vanishes we have ∞^1 such families. The surfaces naturally group themselves in three categories:

1°. Lie's quadrics:

$$Ayz + Bzx + Cxy + Lx + My + Nz = 0.$$

2°. The cubic surfaces:

$$A + Bx + Cy + Dz + Exz + Fxy + Gyz + Hxyx = 0.$$

3°. The cubic and quadric surfaces:

$$\begin{aligned} Axyz + Bxz + Cyz + Dxy + Ex \ + Fz \ &= 0, \\ A + By \ + Cx + Dz \ + Eyz + Fxy &= 0. \end{aligned}$$

To these should also be added the surfaces obtained by transforming these three types by the transformation (37).

The first two categories admit of an involutory transformation

$$x = \frac{\lambda}{x_1}, \quad y = \frac{\mu}{y_1}, \quad z = \frac{\nu}{z_1}$$

which leaves any given surface invariant. They have also a center of involution. The third category admits of no such transformation, and no center of involution exists. The determinant Δ has the following values:

1°. $\Delta_1 = (AL + BM - CN)^2 - 4LMAB.$

2°. $\Delta_2 = (GB + EC - FD - AH)^2 - 4(BC - AF)(GE - HD)$
$(AEGF = BCDH).$

3°. $\Delta_3 = (DF - CE)^2.$

The vanishing of these invariants will be the necessary and sufficient conditions that the respective surfaces shall possess ∞^1 families of curves of the same species.

We note finally that any translation-surface of the form

$$f(e^x, e^y, e^z) = 0$$

gives rise to an algebraic surface having at least two families of curves of the same species. We note also all the algebraic surfaces conjugate to a tetrahedral complex. Such surfaces could be constructed if we knew how to find all the algebraic curves conjugate to such a complex, but only a few cases are known. (See S. Lie, "Berührungstransformationen," pp. 393, 394.) They contain two families of curves of the same species.

WEST VIRGINIA UNIVERSITY, *November* 1, 1909.